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Matrix Continued Fractions

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A matrix continued fraction is defined and used for the approximation of a function $\mathbb{F}$ known as a power series in $1/z$ with matrix coefficients $p \times q$, or equivalently by a matrix of functions holomorphic at infinity. It is a generalization of P-fractions, and the sequence of convergents converges to the given function. These convergents have as denominators a matrix, the columns of which are orthogonal with respect to the linear matrix functional associated to $\mathbb{F}$. The case where the algorithm breaks off is characterized in terms of $\mathbb{F}$. © 1999 Academic Press

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1. INTRODUCTION

A definition of a matrix continued fraction will be given for matrices of any size $p \times q$. The definition is used to define an extension of the Euler–Jacobi–Perron algorithm to the matrix case. When $\mathbb{F}$ is a vector, the method is already known, and studied either in the framework of number theory ($\mathbb{F}$ is a vector of $\mathbf{N}^p$ and the approximations are simultaneous approximations of the p given numbers, see for example [7]) or in the framework of analytic functions (i.e., $\mathbb{F}$ is a vector of analytic functions at infinity for [5], in a neighbourhood of zero for [2]). The two last authors have given non-equivalent definitions, apart from the transformation $z \to 1/z$. This can be seen from the fact that the interruption phenomenon

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is completely characterized in [5], and not in [2]. The two algorithms in [7] and [5] are exactly the same, i.e., the Euler–Jacobi–Perron algorithm, i.e., the extension of Euclid's algorithm for numbers or for polynomials, and, of course, give the same recurrence relations for the numerators and denominators of the approximants (xQ_n in terms of $Q_{n+1}, \cdots, Q_{n-p}$ in our terminology).

The question under study here is the matrix case of size $p \times q$ from which the vector case [5] can be recovered for $q=1$.

Starting from a function $\mathbb{F}$ expanded in a power series in $1/z$ with matrix coefficients of size $p \times q$, a matrix continued fraction is associated, generalizing the P-fractions (or Euclid's algorithm !). The convergents converge to the given function $\mathbb{F}$ with respect to the valuation norm (see Section 4). So an approximation is obtained which can be called a Padé–Hermite approximation in the following sense: $\mathbb{F} = (f_{i,j})$, $i = 1, ..., p$, $j = 1, ..., q$ is given, for each n two matrices $\mathbb{Q}_n$, $\mathbb{P}_n$ are found; $\mathbb{Q}_n$ is a square invertible matrix of size $q \times q$, $\mathbb{P}_n$ is a $p \times q$ matrix and all the terms of the matrix $\mathbb{F} - \mathbb{P}_n \mathbb{Q}_n^{-1}$ satisfy (Theorem 1)

$$(\mathbb{F} - \mathbb{P}_n \mathbb{Q}_n^{-1})_{i,j} = O(1/z^{n_i + m_j + 1}), \qquad i = 1, ..., p; \quad j = 1, ..., q,$$

where $\bar{n} = (n_1, ..., n_p)$ and $\bar{m} = (m_1, ..., m_q)$ are the regular multi-indices ([6], [9] and Section 6) associated to n. It must be emphasised that the Padé–Hermite approximation usually used ([4]) would have given

$$\mathbb{F} - \mathbb{P}_n \mathbb{Q}_n^{-1} = O(1/z^k)\, R(1/z),$$

where R is an analytic function, and with k linked to n. This last notion is less precise than what is found here.

As any rational fraction, $\mathbb{P}_n \mathbb{Q}_n^{-1}$ can be represented by several pairs $\mathbb{P}_n$, $\mathbb{Q}_n$. For one of these, $\mathbb{Q}_n$ has as columns the vectors, orthogonal with respect to the matrix functional associated to $\mathbb{F}$ [9, Theorem 2].

So, in the case of matrices, a natural extension of the scalar Padé approximation ([1]) for $p = q = 1$, or the vector Padé approximation [5, 10] for $q = 1$ has been found, including all the aspects: rational approximation, orthogonal polynomials [9], continued fraction.

The last part, following [5] for the vector case (and Euler for the scalar case) looks at the problem of the interruption of the algorithm. This case is characterized in terms of $\mathbb{F}$ by the following: the algorithm is interrupted if and only if there exists a $q \times (q+p)$ matrix $C = (\alpha, \beta)$ (α and β are respectively of size $q \times q$ and $p \times q$) of maximum rank q such that

$$\det \left[C \begin{pmatrix} I_q \\ \mathbb{F} \end{pmatrix} \right] = \det(\alpha + \beta \mathbb{F}) = 0.$$

The classical cases ($p = q = 1$ the function is rational, and $q = 1$ the p scalar functions are linked by a linear relation with polynomial coefficients) are recovered.

2. MATRIX FRACTION

A part of this study has been used for the complete study of a particular case of a continued fraction [8]. We are interested in a ratio of matrices $K \cdot H^{-1}$, $K \in \mathscr{M}_{p,q}$, $H \in \mathscr{M}_{q,q}$ and as usual for rational numbers, an equivalence relation is defined in the set $\mathscr{M}_{p+q,q}$ which is in fact the set of the pairs (H, K)

$$A, A' \in \mathscr{M}_{p+q,q}, \qquad A \sim A' : \exists C \in \mathscr{M}_{q,q}, \qquad \det C \neq 0, \qquad A' = AC$$

Let $\mathscr{G}_{p,q}$ be the set of the equivalent classes of matrices (Grassmann space), then operations are defined in $\mathscr{G}_{p,q}$ through the canonical injection from $\mathscr{M}_{p,q}$ to $\mathscr{G}_{p,q}$. Denote by I_q the unit matrix of size $q \times q$, then

$$\pi : \mathscr{M}_{p,q} \to \mathscr{M}_{p+q,q} \to \mathscr{G}_{p,q}$$

$$A \to \begin{pmatrix} I_q \\ A \end{pmatrix} \to \mathscr{C}l \begin{pmatrix} I_q \\ A \end{pmatrix}.$$

Operations are defined in $\mathscr{G}_{p,q}$ through this canonical injection. Namely, we get for the addition

$$\mathscr{C}l \begin{pmatrix} C \\ A \end{pmatrix} + \mathscr{C}l \begin{pmatrix} D \\ B \end{pmatrix} = \mathscr{C}l \begin{pmatrix} I_q \\ AC^{-1} + BD^{-1} \end{pmatrix},$$

$$\mathscr{C}l \begin{pmatrix} I_q \\ A \end{pmatrix} + \mathscr{C}l \begin{pmatrix} D \\ B \end{pmatrix} = \mathscr{C}l \begin{pmatrix} D \\ AD + B \end{pmatrix}.$$

Canonically, to a transformation w in $\mathscr{M}_{p,q}$ a transformation $\tilde{w}$ in $\mathscr{G}_{p,q}$ is associated (if it exists) such that

$$\tilde{w}\pi(A) = \pi w(A)$$

We now define what will be used as a quotient in the space $\mathscr{M}_{p,q}$ and will be denoted by $1/Z = T(Z)$.

Let $\tilde{T}$, defined on $\mathscr{M}_{p+q,q}$, be the permutation of the rows which puts the last row at the first place. T is defined from $\mathscr{M}_{p,q}$ to $\mathscr{M}_{p,q}$ and by a straightforward computation we obtain the direct definition of T as

$$\text{if } B = \begin{pmatrix} b_{1,1} & \cdots & b_{1,q} \\ & \cdots & \\ b_{p,1} & \cdots & b_{p,q} \end{pmatrix}, \qquad B \to \begin{pmatrix} I_q \\ B \end{pmatrix} \to \tilde{T}\begin{pmatrix} I_q \\ B \end{pmatrix} = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \to T(B) = B_2 B_1^{-1}$$

$$\text{where } B_1 = \begin{pmatrix} b_{p,1} & \cdots & & b_{p,q} \\ 1 & \cdots & & 0 \\ 0 & \ddots & & 0 \\ 0 & \cdots & 1 & 0 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ b_{1,1} & \cdots & & b_{1,q} \\ & \cdots & & \\ b_{p-1,1} & \cdots & & b_{p-1,q} \end{pmatrix}.$$

So, in explicit form, as a transformation of $\mathcal{M}_{p,q}$, we get, if $b_{p,q} \neq 0$

$$T(B) = \frac{1}{b_{p,q}} \begin{pmatrix} 1 & -b_{p,1} & \cdots & -b_{p,q-1} \\ b_{1,q} & b_{1,1}b_{p,q} - b_{1,q}b_{p,1} & \cdots & b_{1,q-1}b_{p,q} - b_{1,q}b_{p,q-1} \\ \cdots & \cdots & & \\ b_{p-1,q} & b_{p-1,1}b_{p,q} - b_{p-1,q}b_{p,1} & \cdots & b_{p-1,q-1}b_{p,q} - b_{p-1,q}b_{p,q-1} \end{pmatrix} \tag{1}$$

It is useful to have also explicitly the inverse application $T^{-1}(A)$ for a $p \times q$ matrix A, if $a_{1,1} \neq 0$

$$T^{-1}(A) = \frac{1}{a_{1,1}} \begin{pmatrix} a_{2,2}a_{1,1} - a_{1,2}a_{2,1} & \cdots & a_{2,q}a_{1,1} - a_{1,q}a_{2,1} & a_{2,1} \\ \cdots & \cdots & & \\ a_{p,2}a_{1,1} - a_{1,2}a_{p,1} & \cdots & a_{p,q}a_{1,1} - a_{1,q}a_{p,1} & a_{p,1} \\ -a_{1,2} & \cdots & -a_{1,q} & 1 \end{pmatrix}. \tag{2}$$

It is a direct extension of the quotient defined for vectors in [5], i.e., for the case $q = 1$

$$T: \begin{pmatrix} b_1 \\ \vdots \\ b_p \end{pmatrix} \to \frac{1}{b_p} \begin{pmatrix} 1 \\ b_1 \\ \vdots \\ b_{p-1} \end{pmatrix}.$$

Because $T^{p+q} = I$, T can be considered as a partial quotient. Moreover, if the matrix A is square $p \times p$, then $T^p(A) = A^{-1}$. $\tilde{T}$ is also the left multiplication by the matrix of permutation J,

$$\tilde{T}\begin{pmatrix} H \\ K \end{pmatrix} = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 1 & \cdots & & 0 \\ 0 & \ddots & & 0 \\ 0 & \cdots & 1 & 0 \end{pmatrix} \begin{pmatrix} H \\ K \end{pmatrix} = J \begin{pmatrix} H \\ K \end{pmatrix}.$$

For an arbitrary given matrix B, let us now consider the homographic application $w(Z)=1/(B+Z)=T(B+Z)$. The corresponding application in $\mathscr{M}_{p+q,q}$ and $\mathscr{G}_{p,q}$ is the left multiplication by a matrix W,

$$\tilde{T}\begin{pmatrix} I_q \\ B+Z \end{pmatrix} = J\begin{pmatrix} I_q \\ B+Z \end{pmatrix}$$

$$= J\begin{pmatrix} I_q & 0 \\ B & I_p \end{pmatrix}\begin{pmatrix} I_q \\ Z \end{pmatrix} = W\begin{pmatrix} I_q \\ Z \end{pmatrix}.$$

More explicitly

$$W = J\begin{pmatrix} I_q & 0 \\ B & I_p \end{pmatrix} = \begin{pmatrix} b_{p,1} & \cdots & b_{p,q} & 0 & \cdots & & 1 \\ 1 & \cdots & & & & & \\ \vdots & \ddots & & & & & \\ & \cdots & 1 & 0 & \cdots & & 0 \\ b_{1,1} & \cdots & b_{1,q} & 1 & \cdots & & 0 \\ \vdots & & \vdots & & \ddots & & \vdots \\ b_{p-1,1} & \cdots & b_{p-1,q} & 0 & \cdots & 1 & 0 \end{pmatrix},$$

$$\det(W) = (-1)^{p+q+1} \tag{3}$$

3. MATRIX CONTINUED FRACTION

For a family of $p \times q$ matrices B_i, the preceding quotient allows us to consider matrix continued fractions

$$\cfrac{1}{B_1 + \cfrac{1}{B_2 + \cfrac{1}{\ddots}}},$$

where $B_i \in \mathscr{M}_{p,q}$ and $1/B = T(B)$. We will always suppose it is possible to form $T(B)$, i.e., $b_{p,q} \neq 0$ and will study the convergents of this continued fraction

$$\Pi_n = \cfrac{1}{B_1 + \cfrac{1}{B_2 + \cfrac{1}{\ddots + \cfrac{1}{B_n}}}}.$$

DEFINITION 1. Continued fractions are called true if all the elements B_k satisfy

(i) $b_{p,q}(z)$ is a polynomial of degree $s_k \geqslant 1$;

(ii) $b_{p,1}(z), ..., b_{p,q-1}(z)$ and $b_{1,q}(z), ..., b_{p-1,q}(z)$ are polynomials of degree smaller than $s_k - 1$;

(iii) all $b_{i,j}(z), i = 1, ..., p-1; j = 1, ..., q-1$ are polynomials of degree smaller than $s_k - 2$.

DEFINITION 2. A true continued fraction is called a regular continued fraction if for all k, $s_k = 1$.

In the case of a regular continued fraction the $B_k(z)$ are of the following kind, with α, β, γ, δ constants, and the α nonzero:

$$B_k(z) = \begin{pmatrix} 0 & & 0 & \gamma_{k,1} \\ & \cdots & & \cdots \\ 0 & & 0 & \gamma_{k,p-1} \\ \delta_{k,1} & \cdots & \delta_{k,q-1} & \alpha_k z + \beta_k \end{pmatrix}. \tag{4}$$

We will show that for the considered functions the continued fraction is always a true continued fraction, and it will be assumed to be a regular one.

4. THE SET OF THE CONSIDERED FUNCTIONS

As usual for Padé approximation in the neighbourhood of infinity, the functions are of the type

$$f \in \mathbb{C}[[1/z]], \qquad f(z) = \sum_{k=0}^{\infty} \frac{c_k}{z^{k+1}}$$

where c_k are complex constants. In this vector space of formal power series in $1/z$ without constant term, the following norm is defined

$$v = \inf\{k, c_k \neq 0\}, \qquad \|f\| = e^{-v}$$

and satisfies $\|f + g\| \leqslant \max(\|f\|, \|g\|)$. In the following, convergence is to be understood with respect to this norm.

For functions in $\mathbb{C}[[1/z]]$, $1/f$ is clearly the sum of a polynomial of degree at least one and a series of the same kind, so f is classically approximated by continued fractions [3, 11]

$$\cfrac{1}{b_1(z)+\cfrac{1}{b_2(z)+\cfrac{1}{\ddots}}},$$

where $b_k(z)$ are polynomials. The same is done for a matrix $\mathbb{F}(z)$ with elements in $\mathbb{C}[[1/z]]$, and the convergence is studied for the same norm on each component.

5. THE ALGORITHM

A description of the algorithm to obtain $\mathbb{F}(z)$ as a continued fraction is given; it is a direct generalization of Euclid's algorithm, or also of P-fractions [3, 5, 11]. The notation $P[A]$ means the polynomial part of A, i.e., each term is the polynomial part of the corresponding term of A if A is a matrix, and $1/A$ means $T(A)$

$$\mathbb{F}=\mathbb{F}^0=\frac{1}{P^1+\mathbb{F}^1}, \qquad P^1+\mathbb{F}^1=T^{-1}(\mathbb{F})$$

$$\mathbb{F}^n=\frac{1}{P^{n+1}+\mathbb{F}^{n+1}}, \qquad P^{n+1}+\mathbb{F}^{n+1}=T^{-1}(\mathbb{F}^n)$$

We obtain, formally,

$$\mathbb{F}=\cfrac{1}{P^1+\cfrac{1}{P^2+\cfrac{1}{\ddots}}}.$$

In $\mathscr{M}_{p+q,q}$ (and in $\mathscr{G}_{p,q}$), it follows, with W_n the $(p+q)\times(p+q)$ matrix associated to the homographic application $w_n(Z)=T(P^n+Z)$,

$$\begin{aligned}
&\begin{pmatrix} I_q \\ \mathbb{F} \end{pmatrix}=\begin{pmatrix} I_q \\ \mathbb{F}^0 \end{pmatrix}=\tilde{T}\begin{pmatrix} I_q \\ P^1+\mathbb{F}^1 \end{pmatrix}, \qquad P^1=P[T^{-1}(\mathbb{F}^0)]\\
&\begin{pmatrix} I_q \\ \mathbb{F}^n \end{pmatrix}=\tilde{T}\begin{pmatrix} I_q \\ P^{n+1}+\mathbb{F}^{n+1} \end{pmatrix}=W_{n+1}\begin{pmatrix} I_q \\ \mathbb{F}^{n+1} \end{pmatrix}\\
&\begin{pmatrix} I_q \\ \mathbb{F} \end{pmatrix}=\begin{pmatrix} I_q \\ \mathbb{F}^0 \end{pmatrix}=M_n\begin{pmatrix} I_q \\ \mathbb{F}^n \end{pmatrix}\\
&\quad M_0=I_{p+q}, \qquad M_n=M_{n-1}W_n, \qquad M_n=W_1\cdots W_n.
\end{aligned} \tag{6}$$

An expression of the convergent of order n is obtained;

$$\begin{pmatrix} I_q \\ \Pi_n \end{pmatrix} = M_n \begin{pmatrix} I_q \\ 0 \end{pmatrix}. \tag{7}$$

It follows that, if y_n^k, $k=1, \cdots, p+q$ are the columns of M_n,

$$M_n = (y_n^1, ..., y_n^{p+q}), \qquad \text{then} \quad \begin{pmatrix} I_q \\ \Pi_n \end{pmatrix} = (y_n^1, ..., y_n^q).$$

6. PROPERTIES OF THE CONTINUED FRACTION

We study the properties of the continued fraction associated to a $p \times q$ matrix of functions of $\mathbb{C}[[1/z]]$. We first recall the definition of a regular multi-index ([6])

DEFINITION 3. A multi-index $\bar{k} = (k_1, ..., k_d) \in \mathbb{Z}^d$ is called regular if

$$k_1 \geqslant k_2 \geqslant \cdots \geqslant k_d \geqslant k_1 - 1$$

Any integer n defines only one regular multi-index of size d such that $\sum_1^d n_i = n$: if $n = vd + k$ with $0 \leqslant k < d$, then $(n_1, ..., n_d) = (v+1, ..., v+1, v, ..., v)$ i.e., k terms $v+1$.

LEMMA 1. *If $\mathbb{F}$ is a $p \times q$ matrix of functions of $\mathbb{C}[[1/z]]$, then the continued fraction associated to $\mathbb{F}$ is a true continued fraction.*

Proof. The proof is a direct consequence of (2).

Let A, written for any $\mathbb{F}^n$ of the algorithm, be a $p \times q$ matrix, with $a_{i,j} \in \mathbb{C}[[1/z]]$ for all i and j. Then $T^{-1}(A)$ is the sum of a matrix polynomial P and of a matrix B where all elements $b_{i,j} \in \mathbb{C}[[1/z]]$:

$$\text{If} \quad a_{1,1} = \frac{\alpha_s}{z^s} + \cdots, \qquad \text{then} \quad \frac{1}{a_{1,1}} = \frac{z^s}{\alpha_s}(1 + \text{decreasing powers of } z), \quad s \geqslant 1.$$

Thus $(T^{-1}(A))_{p,q}$ is the sum of a polynomial of degree s and of $b_{p,q}$ a function of $\mathbb{C}[[1/z]]$.

$(T^{-1}(A))_{i,j}$, for $i = p$ or $j = q$ is a quotient $a_{i,j}/a_{1,1}$ so is the sum of a polynomial of degree at most $s-1$ and of a function of $\mathbb{C}[[1/z]]$. Similarly $(T^{-1}(A))_{i,j}$ for $i \neq p$ and $j \neq q$ is the sum of a polynomial of degree at most $s-2$ and a function of the right kind. So $\mathbb{F}$ is associated to a true continued fraction.

Moreover, if $a_{1,1}$ has a nonzero first term α/z, i.e., $s=1$, then P has the form (4). ▌

Two sequences of multi-indices, of sizes respectively q and p, are defined in the following way. For any integer k, let s_k be the degree of $(P_k)_{p,q}$ ($s_k \geqslant 1$) and $k = vq + \mu$, $0 \leqslant \mu < q$, then $\bar{m}^k = (m_1^k, ..., m_q^k)$, where

$$\begin{aligned} m_j^k &= s_j + s_{q+j} + \cdots + s_{vq+j}, & j = 1, ..., \mu \\ &= s_j + s_{q+j} + \cdots + s_{(v-1)q+j}, & j = \mu + 1, ..., q. \end{aligned}$$

Now do the same to obtain a sequence $(\bar{n}^k)_{k>0}$, p replacing q. Then the following theorem is obtained for the approximation of $\mathbb{F}$ by the convergents of the continued fraction

THEOREM 1. *Let k be any integer, defining the multi-indices $\bar{m}^k = (m_1^k, ..., m_q^k)$ and $\bar{n}^k = (n_1^k, ..., n_p^k)$. If Π_k is the convergent of order k of $\mathbb{F}$, then the order of approximation is*

$$(\mathbb{F} - \Pi_k)_{i,j} = O(1/z^{n_i^k + m_j^k + 1}).$$

This will be denoted by

$$\mathbb{F} - \Pi_k = O(1/z^{\bar{n}^k + \bar{m}^k + 1}). \tag{8}$$

Proof. The proof is by recurrence.

Let us consider the case $k = 1$:

$$\mathbb{F}^0 = (a_{i,j})_{i=1,...,p;\, j=1,...,q} = \frac{1}{P^1 + \mathbb{F}^1}$$

$$P^1 + \mathbb{F}^1 = \begin{pmatrix} a_{2,2} - a_{2,1}\dfrac{a_{1,2}}{a_{1,1}} & \cdots & a_{2,q} - a_{2,1}\dfrac{a_{1,q}}{a_{1,1}} & \dfrac{a_{2,1}}{a_{1,1}} \\ \vdots & & \vdots & \\ -\dfrac{a_{1,2}}{a_{1,1}} & \cdots & -\dfrac{a_{1,q}}{a_{1,1}} & \dfrac{1}{a_{1,1}} \end{pmatrix}$$

$$P^1 = \begin{pmatrix} & \cdots & & P\left[\dfrac{a_{2,1}}{a_{1,1}}\right] \\ \vdots & & & \vdots \\ -P\left[\dfrac{a_{1,2}}{a_{1,1}}\right] & \cdots & -P\left[\dfrac{a_{1,q}}{a_{1,1}}\right] & P\left[\dfrac{1}{a_{1,1}}\right] \end{pmatrix}.$$

We now compare $\Pi_1 = T(P^1)$ and $\mathbb{F} = \mathbb{F}^0$ term by term. We write

$$a_{1,1} = \frac{1}{A_{1,1}}, \qquad s_1 = \deg(P[A_{1,1}]).$$

So it follows

$$\begin{aligned}
(\mathbb{F} - \Pi_1)_{1,1} &= a_{1,1} - \frac{1}{P[1/a_{1,1}]} = \frac{A_{1,1} + O(1/z) - A_{1,1}}{A_{1,1} P[A_{1,1}]} \\
&= O(1/z^{1+2s_1}) \\
(\mathbb{F} - \Pi_1)_{1,j} &= a_{1,j} - \frac{P[a_{1,j}/a_{1,1}]}{P[1/a_{1,1}]} = a_{1,j} - \frac{P[a_{1,j} A_{1,1}]}{P[A_{1,1}]} \\
&= \frac{a_{1,j}(A_{1,1} + O(1/z)) - (a_{1,j} A_{1,1} + O(1/z))}{P[A_{1,1}]} \\
&= O(1/z^{1+s_1}), \qquad j = 2, ..., q \\
(\mathbb{F} - \Pi_1)_{i,1} &= a_{i,1} - \frac{P[a_{i,1}/a_{1,1}]}{P[1/a_{1,1}]} = O(1/z^{1+s_1}), \qquad i = 2, ..., p \\
(\mathbb{F} - \Pi_1)_{i,j} &= O(1/z), \qquad i \neq 1, \; j \neq 1.
\end{aligned}$$

At step $k+1$, we have, replacing the continued fraction from P^2 to P^{k+1} by $\mathbb{F}^1 - \varepsilon$,

$$\mathbb{F}^0 = \frac{1}{P^1 + \mathbb{F}^1}, \qquad \mathbb{F}^1 - \left(\frac{1}{P^2} + \cdots + \frac{1}{P^{k+1}} \right) = (\varepsilon_{i,j})_{i=1,...,p;\; j=1,...,q}$$

$$\mathbb{F}^0 - \Pi_{k+1} = \frac{1}{P^1 + \mathbb{F}^1} - \frac{1}{P^1 + \mathbb{F}^1 - \varepsilon}.$$

Comparing term by term, it follows, with the same computations as before:

$$\begin{aligned}
(\mathbb{F}^0 - \Pi_{k+1})_{1,1} &= \varepsilon_{p,q} O(1/z^{2s_1}) \\
(\mathbb{F}^0 - \Pi_{k+1})_{1,j} &= \varepsilon_{p,j-1} O(1/z^{s_1}), \qquad j = 2, ..., q \\
(\mathbb{F}^0 - \Pi_{k+1})_{i,1} &= \varepsilon_{i-1,q} O(1/z^{s_1}), \qquad i = 2, ..., p \\
(\mathbb{F}^0 - \Pi_{k+1})_{i,j} &= \varepsilon_{i-1,j-1}, \qquad i = 2, ..., p, \quad j = 2, ..., q.
\end{aligned}$$

This gives the recurrence relation for the order of approximation denoted by $(n^k_{i,j}(s_1, ..., s_k))$, $i = 1, ..., p$; $j = 1, ..., q$,

$$\mathbb{F} - \Pi_k = (O(1/z^{n^k_{i,j}}))_{i=1,\ldots,p;\ j=1,\ldots,q}$$

$$n^{k+1}_{1,1}(s_1, s_2, \ldots, s_{k+1}) = n^k_{p,q}(s_2, \ldots, s_{k+1}) + 2s_1$$

$$n^{k+1}_{1,j}(s_1, s_2, \ldots, s_{k+1}) = n^k_{p,j-1}(s_2, \ldots, s_{k+1}) + s_1, \qquad j = 2, \ldots, q$$

$$n^{k+1}_{i,1}(s_1, s_2, \ldots, s_{k+1}) = n^k_{i-1,q}(s_2, \ldots, s_{k+1}) + s_1, \qquad i = 2, \ldots, p$$

$$n^{k+1}_{i,j}(s_1, s_2, \ldots, s_{k+1}) = n^k_{i-1,j-1}(s_2, \ldots, s_{k+1}), \qquad i = 2, \ldots, p, \quad j = 2, \ldots, q$$

and from the initial condition ($n^0_{i,j} = 1$ for all i and j) it is easy to conclude, with the previous definitions of $\bar{m}^k = (m^k_1, \ldots, m^k_q)$ and $\bar{n}^k = (n^k_1, \ldots, n^k_p)$ that

$$n^k_{i,j} = n^k_i + m^k_j + 1, \qquad i = 1, \ldots, p; \quad j = 1, \ldots, q,$$

which is the required result. ∎

In the case where the continued fraction associated to $\mathbb{F}$ is not only a true continued fraction, but also a regular one, we have $s_k = 1$ for all k, and so the multi-indices $\bar{m}^k$ and $\bar{n}^k$ are the regular multi-indices of sizes respectively q and p defined by k, i.e., if $k = \nu q + \mu$, $0 \leqslant \mu < q$, then $\bar{m}^k = (\nu+1, \ldots, \nu+1, \nu, \ldots, \nu)$ (μ terms equal to $\nu + 1$), and similarly for the $\bar{n}$.

The following theorem for the convergence of the continued fraction is now obtained.

THEOREM 2. *Every true continued fraction converges to some matrix.*

Proof. The convergence of the sequence of convergents Π_n is equivalent to the convergence of the series $\sum_{k \geqslant 0} (\Pi_{k+1} - \Pi_k)$. With the considered norm, the necessary condition

$$\|\Pi_{k+1} - \Pi_k\| \to 0$$

is also sufficient. As Π_k is the convergent of the matrix Π_N, $N \geqslant k$, by the preceding theorem

$$(\Pi_{k+1} - \Pi_k)_{i,j} = O(1/z^{n^k_i + m^k_j + 1}), \qquad i = 1, \ldots, p; \quad j = 1, \ldots, q$$

and the result is obtained. ∎

The continued fraction obtained from a given function is unique, as described in the following

THEOREM 3. *If the limit of a continued fraction is decomposed into a continued fraction by the previous algorithm, then the same fraction is obtained.*

Proof.

$$\mathbb{F}=\frac{1}{P_1+\cdots}=\frac{1}{P_1+\mathbb{F}^1}.$$

With the (not confusing) notation T, we get

$$(T^{-1}(\mathbb{F}))_{p,q}=\frac{1}{(\mathbb{F})_{1,1}}=\frac{1}{\dfrac{1}{(P_1+\mathbb{F}^1)_{p,q}}}=(P_1+\mathbb{F}^1)_{p,q}$$

and so $P[T^{-1}(\mathbb{F})]_{p,q}=(P_1)_{p,q}$, and so on. ∎

THEOREM 4. *Suppose the continued fraction deduced from $\mathbb{F}$ is a regular one. Denote the columns of the matrix M_n by y_n^k, $k=1, ..., p+q$, then the convergent of the continued fraction, written in $\mathcal{G}_{p,q}$ is*

$$\mathcal{C}l\begin{pmatrix} I_q \\ \Pi_n \end{pmatrix}=\mathcal{C}l(y_n^1, y_n^2, ..., y_n^q)=\mathcal{C}l(y_n^1, y_{n+1}^1, ..., y_{n+q}^1).$$

The sequence $(y_n^1)_{n\geqslant 0}$ of vectors of size $p+q$ satisfies the following recurrence relation, for $n+q\geqslant 1$ (where the constants α, β, γ, δ are defined in (4))

$$\begin{aligned} y_{n+q}^1 = {} & y_{n+q-1}^1\,\delta_{n+q,1}+\cdots+y_{n+1}^1\,\delta_{n+2,q-1}+y_n^1(\alpha_{n+1}z+\beta_{n+1}) \\ & +y_{n-1}^1\gamma_{n+1,p-1}+\cdots+y_{n-p+1}^1\gamma_{n+1,1}+y_{n-p}^1 \end{aligned} \tag{9}$$

with the initial conditions $y_n^1=0$, $n<0$, $y_0=(1, 0, ..., 0)^t$.

Proof. The convergent of order n is

$$\begin{pmatrix} I_q \\ \Pi_n \end{pmatrix}=M_n\begin{pmatrix} I_q \\ 0 \end{pmatrix}, \qquad M_{n+1}=M_n W_{n+1}, \qquad \text{i.e.,}$$

$$(y_{n+1}^1, y_{n+1}^2, ..., y_{n+1}^{p+q})=(y_n^1, y_n^2, ..., y_n^{p+q})\, W_{n+1}$$

$$W_n=\begin{pmatrix} \delta_{n,1} & \cdots & \delta_{n,q-1} & \alpha_n z+\beta_n & 0 & \cdots & 1 \\ 1 & \cdots & & & \vdots & & \\ \vdots & \ddots & & & \vdots & \vdots & \\ \vdots & & \ddots & & \vdots & & \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \gamma_{n,1} & 1 & \cdots & 0 \\ \vdots & & & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \gamma_{n,p-1} & 0 & 1 & 0 \end{pmatrix}.$$

So the recurrence relations for the y_n^k are

$$\begin{aligned}
y_{n+1}^1 &= y_n^1\,\delta_{n+1,1} + y_n^2\\
&\vdots\\
y_{n+1}^{q-1} &= y_n^1\,\delta_{n+1,q-1} + y_n^q\\
y_{n+1}^q &= y_n^1(\alpha_{n+1}z+\beta_{n+1}) + y_n^{q+1} + \gamma_{n+1,1}y_n^{q+2} + \cdots + \gamma_{n+1,p-1}y_n^{p+q}\\
y_{n+1}^{q+k} &= y_n^{q+k+1}, \qquad k=1,\ldots,p-1\\
y_{n+1}^{q+p} &= y_n^1.
\end{aligned} \tag{10}$$

Summing the relations, written with the right indices, the recurrence relation for the sequence $(y_n^1)_{n\geqslant 0}$ is found as (9)

$$\begin{aligned}
y_{n+q}^1 = y_{n+q-1}^1\,\delta_{n+q,1} + \cdots + y_{n+1}^1\,\delta_{n+2,q-1} + y_n^1(\alpha_{n+1}z+\beta_{n+1})\\
+ y_{n-1}^1\gamma_{n+1,p-1} + \cdots + y_{n-p+1}^1\gamma_{n+1,1} + y_{n-p}^1
\end{aligned}$$

The initial conditions follow from (6).

The other columns of the convergent $y_n^2, \ldots, y_n^q$ are linear combinations of the y_{n+k}^1

$$\begin{aligned}
y_n^2 &= y_{n+1}^1 - y_n^1\,\delta_{n+1,1}\\
y_n^3 &= y_{n+1}^2 - y_n^1\,\delta_{n+1,2} = y_{n+2}^1 - y_{n+1}^1\,\delta_{n+2,1} - y_n^1\delta_{n+1,2}\\
&\vdots\\
y_n^q &= y_{n+1}^{q-1} - y_n^1\,\delta_{n+1,q-1} = y_{n+q-1}^1 - y_{n+q-2}^1\delta_{n+q-1,1} - \cdots - y_n^1\delta_{n+1,q-1}.
\end{aligned}$$

In matrix form, we get

$$(y_n^1, y_n^2, \ldots, y_n^q) = (y_n^1, y_{n+1}^1, \ldots, y_{n+q-1}^1)
\times \begin{pmatrix}
1 & -\delta_{n+1,1} & -\delta_{n+1,2} & \cdots & -\delta_{n+1,q-1}\\
 & 1 & -\delta_{n+2,1} & \cdots & -\delta_{n+2,q-2}\\
 & & \ddots & & \vdots\\
 & & & 1 & -\delta_{n+q-1,1}\\
 & & & & 1
\end{pmatrix}.$$

As the matrix on the right is invertible, it follows that in $\mathcal{G}_{p,q}$ we have

$$\mathcal{C}l\begin{pmatrix} I_q\\ \Pi_n\end{pmatrix} = \mathcal{C}l(y_n^1, y_n^2, \ldots, y_n^q) = \mathcal{C}l(y_n^1, \ldots, y_{n+q}^1)$$

which ends the proof. ∎

The approximant found as $(y_n^1, y_{n+1}^1, ..., y_{n+q}^1)$ is a $(p+q)\times q$ matrix which can be written in two blocks of size respectively $q\times q$ and $p\times q$, as

$$(y_n^1, y_{n+1}^1, ..., y_{n+q}^1)=\begin{pmatrix}\mathbb{Q}_n\\ \mathbb{P}_n\end{pmatrix},$$

$$\mathbb{Q}_n=(Q_n, Q_{n+1}, ..., Q_{n+q}),$$

$$\mathbb{P}_n=(P_n, P_{n+1}, ..., P_{n+q}).$$

Since the sequence $(Q_n)_{n\geqslant 0}$ is a sequence of vector polynomials of size q, which satisfies the recurrence relation (9), we know [9] that it is a matrix-orthogonal sequence with respect to a $(p\times q)$ matrix of linear forms $\Theta=(\Theta_{i,j})$, $i=1, ..., p$; $j=1, ..., q$.

Let $(h_n)_{n\geqslant 0}$ be the canonical basis of $\mathbb{C}[X]^q$ (i.e., for $n=\nu q+\nu_0$, h_n has only the component ν_0 which is nonzero, and is x^ν). The Q_n are defined by the recurrence relation (9) and the initial conditions. Looking at the first column of the product $W_1\cdots W_n$, it is clear that

$$Q_n=c_0h_0+\cdots+c_nh_n, \qquad c_n\neq 0, \quad n=0, ..., q-1.$$

The recurrence relations (9) for the Q_n can be written as

$$(diag\ \alpha_n)\, zQ=AQ=\begin{pmatrix} a_0^{(0)} & \cdots & \cdots & 1 & 0 & 0 & \\ \vdots & a_1^{(0)} & \cdots & \cdots & 1 & 0 & \\ \vdots & & \ddots & & & \ddots & \\ -1 & \cdots & \cdots & & & & \\ 0 & -1 & \cdots & \cdots & & & \\ 0 & 0 & \ddots & & & & \end{pmatrix} Q, \qquad \forall n, \alpha_n\neq 0, \tag{11}$$

where Q is the infinite column vector $(Q_0, Q_1, ...)^t$ (each term being a vector, Q could be written as a scalar matrix $(\infty\times q)$) and A a scalar infinite band matrix with $p+q+1$ diagonals. The two extreme diagonals having nonzero terms, the matrix form is non degenerate ([9]) if and only if all the $\alpha_n\neq 0$, and for all n, Q_n is expanded in the basis $(h_i)_{i\geqslant 0}$ exactly up to the term h_n.

We can now consider again the question of the degree of approximation, not for $\mathbb{F}-\Pi_k$ as in theorem 1 but for the weak approximation $\mathbb{F}\mathbb{Q}_k-\mathbb{P}_k$, which is equivalent to considering each column of the preceding matrix, and in particular the first column $\mathbb{F}Q_k-P_k$.

THEOREM 5. *Suppose the continued fraction is regular. If the index* k *defines the regular multi-index of size* p, $\bar{n} = (n_1, ..., n_p)$, *then*

$$(\mathbb{F}Q_k - P_k)_i = O(1/z^{n_i+1}), \qquad i = 1, ..., p$$
$$(\mathbb{F}Q_k - P_k) = O(1/z^{\bar{n}+1}). \tag{12}$$

Proof. The index k defines also the regular index of size q, $\bar{m} = (m_1, ..., m_q)$ and because Q_k is expanded in the basis $h_0, ..., h_k$, the component i of Q_k is of degree at most m_i for $i = 1, ..., q$.
We have

$$\mathbb{F}Q_k - P_k = (\mathbb{F} - \Pi_k)\, Q_k.$$

Using Theorem 1, it follows for z going to infinity that, for $i = 1, ..., p$,

$$(\mathbb{F}Q_k - P_k(z))_i = \sum_{j=1}^{q} O\left(\frac{1}{z^{n_i+m_j+1}}\right) z^{m_j} = O\left(\frac{1}{z^{n_i+1}}\right),$$

and the conclusion for the weak approximation is obtained. ∎

From this, the approximation of $\mathbb{F}$ is either Π_k or the two matrices $\mathbb{Q}_k$ and $\mathbb{P}_k$, with $\Pi_k = \mathbb{P}_k(\mathbb{Q}_k)^{-1}$ satisfying

$$(\mathbb{F} - \Pi_k)_{i,j} = O(1/z^{n_i+m_j+1}), \qquad i = 1, ..., p, \quad j = 1, ..., q$$
$$\mathbb{F}\mathbb{Q}_k - \mathbb{P}_k = \mathbb{O}(1/z^{\bar{n}+1}),$$

the right hand side of the second formula meaning a matrix of terms $1/z^*$ where the powers of $1/z$ are regular multi-indices on each row and column, decreasing in the columns, increasing in the rows, starting from $\bar{n}$ defined by k in the first column, i.e., writing only the powers of the matrix $\mathbb{O}(1/z^{\bar{n}+1})$ we get, if $k = vp + \mu$, $0 \leqslant \mu < p$

$$\begin{pmatrix} v+1 & \cdots & v+1 & v+2 & v+2 & \cdots \\ v+1 & \cdots & v+1 & v+1 & v+2 & \cdots \\ \vdots & & \vdots & \ddots & \ddots & \\ v & \ddots & v+1 & \cdots & & \\ v & v & v+1 & \cdots & & \end{pmatrix}.$$

So the continued fraction gives rise to a matrix Padé approximant of $\mathbb{F}$.

P_k being a vector polynomial is, from the approximation property, necessarily the polynomial part of $\mathbb{F}Q_k$, and with the notations

$$i = 1, ..., p; \qquad j = 1, ..., q, \qquad f_{i,j} = \sum_{v=0}^{\infty} \frac{f_{i,j}^{v}}{z^{v+1}}, \qquad \Theta_{i,j}(x^v) = f_{i,j}^{v}$$

it follows, each functional acting on x, k defining $(m_1, ..., m_q)$ and $(P_k)_i$ being the component i of P_k [9]

$$i=1, ..., p, \qquad (P_k)_i(z)=\sum_{j=1}^{q} \Theta_{i,j}\left(\frac{(Q_k)_j(x)-(Q_k)_j(z)}{x-z}\right)$$

$$\text{i.e.,} \qquad P_k(z)=\Theta\left(\frac{Q_k(x)-Q_k(z)}{x-z}\right), \qquad \deg(P_k)_i=m_1-1.$$

From this formula or from $\mathbb{P}_k=P[\mathbb{F}\mathbb{Q}_k]$, the degree of the components $(P_k)_i$, $i=1, ..., p$, is known: each $(Q_k)_j$ is of degree m_j for $j=1, ..., q$, so $(P_k)_i$ is the sum of polynomials of degree respectively m_j-1, and so is of degree less than or equal to m_1-1 for all i between 1 and p.

From the results of this part and from [9], it follows in linear system terminology

THEOREM 6. *The continued fraction deduced from $\mathbb{F}$ is regular if and only if $\mathbb{F}$ is a weakly perfect matrix.*

7. WHAT IF THE ALGORITHM IS INTERRUPTED?

The algorithm which constructs the continued fraction from a given matrix function $\mathbb{F}$ is interrupted if it is not possible to compute $T^{-1}(\mathbb{F}^n)$. In the scalar case, this is characteristic of a function $\mathbb{F}$ which is a rational function. In the vector case (i.e., $q=1$), it has been proved [5] that the components of the given function satisfy a linear equation with polynomial coefficients. These results are particular cases of the following necessary and sufficient condition which completely characterizes the interruption phenomenon in the matrix case

THEOREM 7. *The algorithm is interrupted if and only if there exists a $q\times(q+p)$ matrix $C=(\alpha, \beta)$ with polynomial entries, of maximum rank q such that*

$$\det\left(C\begin{pmatrix} I_q \\ \mathbb{F}\end{pmatrix}\right)=\det(\alpha+\beta\mathbb{F})=0. \tag{13}$$

Proof. Let us suppose first that the algorithm is interrupted. It is not possible to compute $T^{-1}(\mathbb{F}^n)$, which is equivalent to $a^n_{1,1}=0$. In $\mathcal{G}_{p,q}$ we have

$$\begin{pmatrix} I_q \\ \mathbb{F}\end{pmatrix}=M_n\begin{pmatrix} I_q \\ \mathbb{F}^n\end{pmatrix}.$$

As $M_n = W_1 \cdots W_n$ and $|\det W_k| = 1$, M_n is a matrix with determinant plus or minus one, all its coefficients are polynomials, and its inverse matrix has the same properties: polynomial coefficients and determinant ± 1. Written in block form, we get

$$\begin{pmatrix} I_q \\ \mathbb{F}^n \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \alpha' & \beta' \end{pmatrix} \begin{pmatrix} I_q \\ \mathbb{F} \end{pmatrix},$$

with $\alpha + \beta\, \mathbb{F}$ invertible. We compute $a^n_{1,1}$ from the formula

$$\mathbb{F}^n = (\alpha' + \beta' \mathbb{F})(\alpha + \beta \mathbb{F})^{-1}.$$

Let us write $\alpha = (\alpha_{i,j})$, $i, j = 1, ..., q$, and similarly for α', which is a $p \times q$ matrix, β_i, $i = 1, ..., q$ and β'_j, $j = 1, ..., p$ the rows of β and β' and $\mathbb{F}_k$, $k = 1, ..., q$ the columns of $\mathbb{F}$. The first row of $\alpha' + \beta' \mathbb{F}$ is

$$(\alpha' + \beta' \mathbb{F})_1 = (\alpha'_{1,1} + \beta'_1 \mathbb{F}_1, \alpha'_{1,2} + \beta'_1 \mathbb{F}_2, ..., \alpha'_{1,q} + \beta'_1 \mathbb{F}_q).$$

The first column of $(\alpha + \beta \mathbb{F})^{-1}$ is, up to the division by the determinant, formed by the cofactors of the first column of $(\alpha + \beta \mathbb{F})^t$. Finally $a^n_{1,1} = 0$ gives

$$\sum_{i=1}^{q} (\alpha'_{1,i} + \beta'_1 \mathbb{F}_i)((\alpha + \beta \mathbb{F})^{-1})_{i,1} = 0$$

$$\begin{vmatrix} \alpha'_{1,1} + \beta'_1 \mathbb{F}_1 & \cdots & \alpha'_{1,q} + \beta'_1 \mathbb{F}_q \\ \alpha_{2,1} + \beta_2 \mathbb{F}_1 & \cdots & \alpha_{2,q} + \beta_2 \mathbb{F}_q \\ \vdots & & \vdots \\ \alpha_{q,1} + \beta_q \mathbb{F}_1 & \cdots & \alpha_{q,q} + \beta_q \mathbb{F}_q \end{vmatrix} = 0.$$

The condition becomes

$$\det \left[(\bar{\alpha}, \bar{\beta}) \begin{pmatrix} I_q \\ \mathbb{F} \end{pmatrix} \right] = 0.$$

The matrix $(\bar{\alpha}, \bar{\beta})$ is the submatrix of M_n^{-1} formed by the rows from index 2 to $q+1$. M_n being invertible, the matrix $(\bar{\alpha}, \bar{\beta})$ is of size $q \times (q+p)$ and of maximum rank q.

Conversely, let us suppose that there exists (α, β) of size $q \times (q+p)$ and of rank q such that

$$\det \left[(\alpha, \beta) \begin{pmatrix} I_q \\ \mathbb{F} \end{pmatrix} \right] = 0,$$

and that the algorithm is not interrupted, i.e., goes on for n as large as necessary, i.e., $\mathbb{Q}_n$ and $\mathbb{P}_n$ are defined for all n. We admit for the moment the following result: If $\det(\alpha+\beta\mathbb{F})=0$, then for all n large enough

$$\det(\alpha\mathbb{Q}_n+\beta\mathbb{P}_n)=0.$$

This result is proved in the last lemma, just below. This is equivalent to saying that for all n large enough

$$(\alpha, \beta)\begin{pmatrix}\mathbb{Q}_n\\ \mathbb{P}_n\end{pmatrix}$$

is of rank strictly smaller than q. But then

$$(\alpha, \beta)\, M_n$$

is also of rank strictly smaller than q and because M_n is invertible, this would mean that the matrix (α, β) is not of maximum rank. Consequently, if

$$\det\left[(\alpha, \beta)\begin{pmatrix}I_q\\ \mathbb{F}\end{pmatrix}\right]=0,$$

with (α, β) of maximum rank, then the algorithm breaks down. ∎

We give now the proof of the lemma already used.
Everywhere in the sequel the determinant will be denoted by

$$\det A=|A|.$$

LEMMA 2. *If* $|\alpha+\beta\mathbb{F}|=0$, *then for* n *large enough*

$$|\alpha\mathbb{Q}_n+\beta\mathbb{P}_n|=0.$$

Proof. Let us denote $\mathbb{F}=(F^1, ..., F^p)^t$, where $(F^i)^t$ is row i of the matrix $\mathbb{F}$. Similarly let $\mathbb{P}_n=(P_n^1, ..., P_n^p)^t$, while $\mathbb{Q}_n=(Q_n, ..., Q_{n+q})$ where Q_n is a column vector of size q.
From the identity

$$\begin{pmatrix}1_q & -\beta\\ 0 & 1_p\end{pmatrix}\begin{pmatrix} & \beta\\ -\mathbb{F} & 1_p\end{pmatrix}=\begin{pmatrix}\alpha+\beta\mathbb{F} & 0\\ -\mathbb{F} & 1_p\end{pmatrix}$$

it follows that

$$|(\alpha+\beta\mathbb{F})\,\mathbb{Q}_n|=\begin{vmatrix}\alpha\mathbb{Q}_n & \beta\\ -\mathbb{F}\mathbb{Q}_n & 1_p\end{vmatrix}.$$

We denote by $\mathscr{A}$ the preceding expression, $\mathbb{Q}_n$ being an invertible matrix, the assumption of the lemma can be written as

$$\mathscr{A}=0. \tag{14}$$

The approximation property $(F^i\mathbb{Q}_n-P^i_n)_j=O(1/z^{[n/p]+1})$, $i=1, ..., p$; $j=1, ..., q$, leads to the following, by expansion of the determinant along the last p rows, and because the degree of each component of αQ_n is $([n/q]+\text{const})$: for $z\to\infty$, n large enough and for any p and q the following expression $\mathscr{B}$

$$\mathscr{B}=\begin{vmatrix}\alpha\mathbb{Q}_n & \beta\\ -(\mathbb{F}\mathbb{Q}_n-\mathbb{P}_n) & O_p\end{vmatrix}\sim\frac{(z^{[n/q]+c})^{\max((q-p),0)}}{(z^{[n/p]+1})^p}\to 0. \tag{15}$$

From the two last equations the polynomial part of $\mathscr{A}-\mathscr{B}$ is zero

$$P[\mathscr{A}-\mathscr{B}]=0$$

In order to write all the terms explicitly, the following is written for $p=3$, and q any integer, but the method leads to the result for all p and q. Moreover, to have a more compact notation

$$\begin{vmatrix}F^1 & a\\ F^2 & b\\ F^3 & c\end{vmatrix}\quad\text{is written for}\quad\begin{vmatrix}\alpha\mathbb{Q}_n & & \beta & \\ F^1 & a & 0 & 0\\ F^2 & 0 & b & 0\\ F^3 & 0 & 0 & c\end{vmatrix},$$

where F^i is a row of size q and a, b, c complex numbers:

$$\begin{aligned}\mathscr{A}-\mathscr{B}&=\mathscr{A}-\begin{vmatrix}-F^1\mathbb{Q}_n+P^1_n & 0\\ -F^2\mathbb{Q}_n+P^2_n & 0\\ P^3 & -1\end{vmatrix}-\begin{vmatrix}-F^1\mathbb{Q}_n+P^1_n & 0\\ -F^2\mathbb{Q}_n+P^2_n & 0\\ -F^3\mathbb{Q}_n & 1\end{vmatrix}\\ &=\mathscr{A}-\begin{vmatrix}-F^1\mathbb{Q}_{\boldsymbol{n}}+P^1_n & 0\\ -F^2\mathbb{Q}_n+P^2_n & 0\\ P^3 & -1\end{vmatrix}-\begin{vmatrix}-F^1\mathbb{Q}_n+P^1_n & 0\\ P^2_n & -1\\ -F^3\mathbb{Q}_n & 1\end{vmatrix}\\ &\quad-\begin{vmatrix}-F^1\mathbb{Q}_n+P^1_n & 0\\ -F^2\mathbb{Q}_n & 1\\ -F^3\mathbb{Q}_n & 1\end{vmatrix}\\ &=\begin{vmatrix}-F^1\mathbb{Q}_n+P^1_n & 0\\ -F^2\mathbb{Q}_n+P^2_n & 0\\ -P^3 & 1\end{vmatrix}+\begin{vmatrix}-F^1\mathbb{Q}_n+P^1_n & 0\\ -P^2_n & 1\\ -F^3\mathbb{Q}_n & 1\end{vmatrix}+\begin{vmatrix}-P^1_n & 1\\ -F^2\mathbb{Q}_n & 1\\ -F^3\mathbb{Q}_n & 1\end{vmatrix}.\end{aligned}$$

Each term is equivalent, for z large, to polynomials of respective degree: $[n/q](q+1)-2[n/p]+c$, $[n/q](q+2)-[n/p]+c'$, $[n/q](q+3)+c''$ where c, c', c'' are some finite constants. The difference between two of these degrees is $[n/q]+[n/p]+c'''$, so for n large enough there does not exist a linear relation between them and the polynomial part of each term is zero, namely

$$P\left[\begin{vmatrix} -P_n^1 & 1 \\ -F^2\mathbb{Q}_n & 1 \\ -F^3\mathbb{Q}_n & 1 \end{vmatrix}\right]=0.$$

Similarly

$$P\left[\begin{vmatrix} -F^1\mathbb{Q}_n & 1 \\ -F^2\mathbb{Q}_n & 1 \\ -P_n^3 & 1 \end{vmatrix}\right]=P\left[\begin{vmatrix} -F^1\mathbb{Q}_n & 1 \\ -P_n^2 & 1 \\ -F^3\mathbb{Q}_n & 1 \end{vmatrix}\right]=0.$$

Returning to $\mathscr{A}-\mathscr{B}$ we get

$$\begin{aligned} P[\mathscr{A}-\mathscr{B}] &= -P\left[\begin{vmatrix} -F^1\mathbb{Q}_n & 1 \\ -P_n^2 & 1 \\ -P_n^3 & 1 \end{vmatrix}+\begin{vmatrix} -P_n^1 & 1 \\ -F^2\mathbb{Q}_n & 1 \\ -P_n^3 & 1 \end{vmatrix}\right. \\ &\quad \left.+\begin{vmatrix} -P_n^1 & 1 \\ -P_n^2 & 1 \\ -F^3\mathbb{Q}_n & 1 \end{vmatrix}-\begin{vmatrix} -P_n^1 & 1 \\ -P_n^2 & 1 \\ -P_n^3 & 1 \end{vmatrix}\right] \\ &= -P\left[\begin{vmatrix} -P_n^1 & 1 \\ -F^2\mathbb{Q}_n & 1 \\ -P_n^3 & 1 \end{vmatrix}+\begin{vmatrix} -F^1\mathbb{Q}_n & 1 \\ -P_n^2 & 1 \\ -P_n^3 & 1 \end{vmatrix}\right] \\ &\quad -P\left[\begin{vmatrix} -P_n^1 & 1 \\ -P_n^2 & 1 \\ -F^3\mathbb{Q}_n+P_n^3 & 0 \end{vmatrix}\right]=0. \end{aligned}$$

As before both polynomials are zero, being of different degrees and zero for z large. By permutation of the indices in the first one, two other terms are zero. Finally, for n large enough, what remains is

$$P[\mathscr{A}-\mathscr{B}]=\begin{vmatrix} -P_n^1 & 1 \\ -P_n^2 & 1 \\ -P_n^3 & 1 \end{vmatrix}=|\alpha\mathbb{Q}_n+\beta\mathbb{P}_n|,$$

from which the required result is obtained

$$\|\alpha \mathbb{Q}_n + \beta \mathbb{P}_n\| = 0.$$

The already known cases, i.e., $p = q = 1$ for the scalar case and $p = 1$ for the vector case are of course recovered. ▮

DEFINITION 4. Functions $f_{i,j}$, $i = 1, ..., p$, $j = 1, ..., q$ are said to be (p, q) dependent if there exists a $q \times (p+q)$ matrix C of polynomials, of maximum rank q such that the matrix $\mathbb{F} = (f_{i,j})_{i=1,...,p,\ j=1,...,q}$ satisfies

$$\det \left[C \begin{pmatrix} I_q \\ \mathbb{F} \end{pmatrix} \right] = 0.$$

This definition is given to have the preceding theorem in the following form: The continued fraction of $\mathbb{F}$ is interrupted if and only if the functions $f_{i,j}$ of $\mathbb{F}$ are (p, q) dependent.

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